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1982 J. Phys. A: Math. Gen. 15 3167

(http://iopscience.iop.org/0305-4470/15/10/020)

J. Phys. A: Math. Gen. 15 (1982) 3167–3172. Printed in Great Britain

Scalar fields and dynamics of the early universe

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Received 18 December 1981, in final form 19 February 1982

Abstract. A theory of a self-interacting scalar field and gravitation is discussed in the context of a Robertson–Walker metric. The calculations are efficiently reformulated using a metric compatible connection with torsion, although the relation to the Brans–Dicke theory is explicitly displayed. New exact solutions are derived and their relevance to recent cosmological models pointed out.

1. Introduction

Model cosmologies are often described (Raychaudhuri 1979) in terms of homogeneous spaces containing simple configurations of matter parametrised in terms of pressure and material density. There has been some interest recently in cosmologies that develop in response to field equations involving scalar fields that interact with themselves and gravitation. Such theories, for example, possess features that have proved of interest in discussing particle creation in the early universe (Parker 1978) and models in which the force of gravitation varies with the age of the cosmos. A number of non-static cosmological solutions have also been discussed (Raychaudhuri 1979, O'Hanlon and Tupper 1972, Blyth and Isham 1975) in the context of the Brans–Dicke scalar–tensor theory.

In a recent paper (Dereli and Tucker 1982) we have indicated how the latter theory may be efficiently reformulated in terms of a metric compatible Lorentz group connection with torsion. By a local rescaling of the metric this reformulation also encompasses the Einstein–Klein–Gordon system. Furthermore the conformal properties of the theory are much simplified in this reformulation.

In this note we present a class of exact non-static solutions to such a theory when a certain scalar self-interaction is included. A similar interaction has featured in a number of recent articles (Davis and Unwin 1981, Ford and Toms 1982) that have discussed broken symmetries in the presence of a background gravitational field.

Some of the solutions below might be interpreted as geometries in the vicinity of the original singularity that is supposed to have triggered the evolution of the universe. It is certainly of interest to examine the cosmic time dependence of the scalar self-interaction, particularly in view of the competing viewpoints concerning the existence of a 'cosmological field' (Davis and Unwin 1981, Christensen and Duff 1980).

0305-4470/82/103167+06$02.00

2. The model

The theory under discussion is generated by varying the action four-form:

$$\Lambda[e^a, \omega_{ab}, \alpha] = \tfrac{1}{2}\alpha^2 R_{ab} \wedge *(e^a \wedge e^b) - \tfrac{1}{2}c\, \mathrm{d}\alpha \wedge *\mathrm{d}\alpha - \lambda\alpha^4 * 1 \tag{1}$$

with respect to the scalar field α, the metric compatible Lorentz group connection one-forms $\omega_{ab} = -\omega_{ba}$ and the orthonormal tetrad e^a. We shall use throughout the language of the exterior calculus and further details of our conventions can be found in Dereli and Tucker (1982). Despite its appearance, for $\lambda = 0$, this is equivalent to the original Brans–Dicke theory. The Brans–Dicke scalar is $\phi = \alpha^2$ and $c = 4\omega + 6$. The canonical Brans–Dicke action follows by solving for the connection ω_{ab} from the equation obtained by varying (1) with respect to this connection:

$$T_a \wedge *(e^a \wedge e^b \wedge e^c) = -2(\mathrm{d}\alpha/\alpha) \wedge *(e^b \wedge e^c). \tag{2}$$

In (1) and (2) the curvature R_{ab} and torsion T^a two-forms are defined by

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{3}$$

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{4}$$

and the space–time metric tensor is

$$g = \eta_{ab} e^a \otimes e^b \tag{5}$$

with $\eta_{ab} = \mathrm{diag}(-, +, +, +)$. $*$ is the Hodge dual map defined with respect to this metric.

The explicit solution for ω_{ab} in terms of α and the torsion-free Christoffel connection $\mathring{\omega}_{ab}$ is

$$\omega_{ab} = \mathring{\omega}_{ab} + (i_b\, \mathrm{d}\alpha/\alpha)e_a - (i_a\, \mathrm{d}\alpha/\alpha)e_b \tag{6}$$

where the contraction operators i_a, $a = 0, 1, 2, 3$, are defined by $i_a(e^b) = \delta_a{}^b$.

In terms of $\mathring{R}_{ab}$ associated with $\mathring{\omega}_{ab}$ the action (1) reduces to

$$\Lambda[e^a, \alpha] = \tfrac{1}{2}\alpha^2 \mathring{R}_{ab} \wedge *(e^a \wedge e^b) - 2\omega\, \mathrm{d}\alpha \wedge *\mathrm{d}\alpha - \lambda\alpha^4 * 1 \tag{7}$$

up to an exact form. In analysing the field equations we find it most advantageous to work from the action (1) rather than (7). Varying (1) with respect to e^a, α and ω_{ab} in turn yields

$$\tfrac{1}{2}\alpha^2 R_{bc} * (e^a \wedge e^b \wedge e^c) = \tau^a + \lambda\alpha^4 * e^a \tag{8}$$

$$\alpha R_{ab} \wedge *(e^a \wedge e^b) = -cd * \mathrm{d}\alpha + 4\lambda\alpha^3 * 1 \tag{9}$$

$$T^a = e^a \wedge (\mathrm{d}\alpha/\alpha) \tag{10}$$

where $\tau^a = -\tfrac{1}{2}c(i^a\, \mathrm{d}\alpha \wedge *\mathrm{d}\alpha + \mathrm{d}\alpha \wedge i^a * \mathrm{d}\alpha)$ are the canonical stress three-forms of the scalar field and (10) is equivalent to (2). Multiplying (8) exteriorly by e_a and comparing with (9) shows that we may replace (9) by the simple equation

$$cd * \mathrm{d}\alpha^2 = 0. \tag{11}$$

Since (10) has the unique solution (6) we may regard (8), (11) and (6) as the basic form equations for the theory. It should be noted that the so-called 'improvement type term' of the Brans–Dicke theory has been assimilated into the curvature associated with the Lorentz connection that solves (10). Furthermore, the theory is locally scale invariant if $c = 0$, in which case, α is arbitrary (Dereli and Tucker 1982).

For $c \neq 0$, we seek cosmological solutions to the coupled field equations, corresponding to a Robertson–Walker metric:

$$g = -\mathrm{d}t \otimes \mathrm{d}t + R^2(t)\{\mathrm{d}\chi \otimes \mathrm{d}\chi + S^2(\chi)[\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\, \mathrm{d}\phi \otimes \mathrm{d}\phi]\} \tag{12}$$

and a scalar field $\alpha = \alpha(t)$. A suitable tetrad would then be

$$e^0 = \mathrm{d}t \qquad e^1 = R(t)\,\mathrm{d}\chi \qquad e^2 = R(t)S(\chi)\,\mathrm{d}\theta \qquad e^3 = R(t)S(\chi)\sin\theta\,\mathrm{d}\phi. \tag{13}$$

In terms of the new variables ρ, σ and μ defined as

$$\rho = \ln|R| \qquad \sigma = \ln|S| \qquad \mu = \ln|\alpha| \tag{14}$$

the connection one-forms, (6), are

$$\omega^0{}_k = (\dot\rho + \dot\mu)e^k \qquad k = 1, 2, 3$$
$$\omega^1{}_2 = -\frac{\sigma'}{R}e^2 \qquad \omega^1{}_3 = -\frac{\sigma'}{R}e^3 \qquad \omega^2{}_3 = -\frac{\cot\theta}{RS}e^3. \tag{15}$$

The associated curvature two-forms are

$$R^0{}_k = A(t)e^0 \wedge e^k \qquad k = 1, 2, 3$$
$$R^1{}_2 = B(t,\chi)e^1 \wedge e^2 \qquad R^1{}_3 = B(t,\chi)e^1 \wedge e^3 \qquad R^2{}_3 = C(t,\chi)e^2 \wedge e^3 \tag{16}$$

where

$$A = (\ddot\rho + \ddot\mu) + \dot\rho(\dot\rho + \dot\mu) \qquad B = (\dot\rho + \dot\mu)^2 - \mathrm{e}^{-2\rho}(\sigma'' + \sigma'^2)$$
$$C = (\dot\rho + \dot\mu)^2 - \mathrm{e}^{-2\rho}(\sigma'^2 - \mathrm{e}^{-2\sigma}). \tag{17}$$

It is also of some interest to compute the Weyl curvature two-forms associated with $\omega^a{}_b$. They are defined by

$$C_{ab} = R_{ab} - \tfrac{1}{2}(e_a \wedge P_b - e_b \wedge P_a) + \tfrac{1}{6}e_a \wedge e_b Q \tag{18}$$

where

$$P_a = i_b R^b{}_a \qquad Q = i_a P^a \tag{19}$$

and we find that

$$\begin{aligned} C_{01} &= \tfrac{1}{3}(B - C)e^0 \wedge e^1 & C_{23} &= \tfrac{1}{3}(C - B)e^2 \wedge e^3 \\ C_{02} &= \tfrac{1}{6}(C - B)e^0 \wedge e^2 & C_{12} &= \tfrac{1}{6}(B - C)e^1 \wedge e^2 \\ C_{03} &= \tfrac{1}{6}(C - B)e^0 \wedge e^3 & C_{13} &= \tfrac{1}{6}(B - C)e^1 \wedge e^3. \end{aligned} \tag{20}$$

Since the torsion vanishes for a constant α, the Riemann–Christoffel curvature forms R_{ab} and the Weyl conformal curvatures $\mathring{C}_{ab}$ associated with $\mathring{\omega}_{ab}$ are obtained from the above formulae by setting μ to a constant. The vanishing of the $\mathring{C}_{ab}$ implies the existence of a metric $\tilde{g}$ conformally related to g that has zero R_{ab}.

With the above ansatz the field equations (8) reduce to

$$C + 2B = \tfrac{1}{2}c\dot\mu^2 + \lambda\,\mathrm{e}^{2\mu} \tag{21}$$
$$C + 2A = -\tfrac{1}{2}c\dot\mu^2 + \lambda\,\mathrm{e}^{2\mu} \tag{22}$$
$$B + 2A = -\tfrac{1}{2}c\dot\mu^2 + \lambda\,\mathrm{e}^{2\mu} \tag{23}$$

and the scalar field equation, (11), to

$$\ddot{\mu}+2\dot{\mu}^2+3\dot{\rho}\dot{\mu}=0. \tag{24}$$

The latter may be integrated immediately to give

$$\dot{\mu}\ \mathrm{e}^{2\mu+3\rho}=\text{constant}. \tag{25}$$

Clearly (22) and (23) imply that $B=C$, so we can simplify (21), (22), (23) to the following:

$$A=-\tfrac{1}{3}c\dot{\mu}^2+\tfrac{1}{3}\lambda\ \mathrm{e}^{2\mu} \qquad B=\tfrac{1}{6}c\dot{\mu}^2+\tfrac{1}{3}\lambda\ \mathrm{e}^{2\mu} \qquad C=B. \tag{26}$$

The last one yields

$$\sigma''+\mathrm{e}^{-2\sigma}=0 \tag{27}$$

which upon integration implies

$$\sigma'^2-\mathrm{e}^{-2\sigma}=\varepsilon \qquad \varepsilon \text{ a constant.} \tag{28}$$

Consequently

$$\sigma''+\sigma'^2=\varepsilon \tag{29}$$

or in terms of S

$$S'^2=1+\varepsilon S^2. \tag{30}$$

This has three well known solutions.

Type 1:	$S=\sin(\sqrt{-\varepsilon}\,\chi)$	for $\varepsilon<0$
Type 2:	$S=\sinh(\sqrt{\varepsilon}\,\chi)$	for $\varepsilon>0$
Type 3:	$S=\chi$	for $\varepsilon=0$.

Then for all ε, $C_{ab}=\mathring{C}_{ab}=0$, even when α and hence μ is not constant and the geometry may be said to be conformally flat.

The remaining equations, (24) and (26), now reduce to

$$\ddot{\rho}+\ddot{\mu}+\dot{\rho}(\dot{\rho}+\dot{\mu})=-\tfrac{1}{3}c\dot{\mu}^2+\tfrac{1}{3}\lambda\ \mathrm{e}^{2\mu} \tag{31}$$

$$(\dot{\rho}+\dot{\mu})^2-\varepsilon\ \mathrm{e}^{-2\rho}=\tfrac{1}{6}c\dot{\mu}^2+\tfrac{1}{3}\lambda\ \mathrm{e}^{2\mu} \tag{32}$$

$$\ddot{\mu}+2\dot{\mu}^2+3\dot{\mu}\dot{\rho}=0 \tag{33}$$

for the two functions $\mu(t)$ and $\rho(t)$.

It is possible at this point to make contact with certain vacuum solutions (O'Hanlan and Tupper 1972) to Brans–Dicke theory with zero λ. For example a type 3 solution ($\varepsilon=0$) with $\lambda=0$ is

$$\rho=\rho_0+m\ \ln t \tag{34}$$

$$\mu=\mu_0+n\ \ln t \tag{35}$$

where ρ_0 and μ_0 are arbitrary constants and

$$1/n=-1\pm(3c/2)^{1/2} \tag{36}$$

$$m=\tfrac{1}{3}(1-2n). \tag{37}$$

In this class c is an arbitrary positive constant. For $\mu=\mu_1 t$ and $\rho=\rho_1 t$ with μ_1, ρ_1

constant there is another solution if $\rho_1 = \frac{2}{3}\mu_1$ and $c = \frac{2}{3}$. These type 3 ($\lambda = 0$) solutions coincide with those found in O'Hanlan and Tupper (1972). If the zero λ condition is relaxed then we have for type 3 the de Sitter solution

$$R(t) = R_0 \exp[\pm(\tfrac{1}{3}\lambda\alpha_0^2)^{1/2} t] \tag{38}$$

$$\alpha = \alpha_0 \tag{39}$$

with R_0, α_0 constants.

The new solutions we find in the case when $\varepsilon \neq 0$, $\lambda \neq 0$ follow from the forms

$$\mu = \ln|\alpha_0 t^n| \tag{40}$$

$$\rho = \ln|R_0 t^m| \tag{41}$$

where α_0 and R_0 are constants. The equations (31), (32), (33) are consistent if $m = -n = 1$ and the constants α_0 and R_0 are related to ε, λ and c by the conditions

$$\alpha_0^2 = c/\lambda \tag{42}$$

$$R_0^2 = -2\varepsilon/c. \tag{43}$$

Then for $c > 0$ we must have $\varepsilon < 0$ (type 1) and $\lambda > 0$. For $c < 0$ then $\varepsilon > 0$ (type 2) and $\lambda < 0$. The type 1 configuration is usually regarded as referring to an action with physical scalar field. In this case if we change to the coordinate $r = \sin(\sqrt{-\varepsilon}\chi)$ this solution may be written as

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \frac{2}{c} t^2 \left(\frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - r^2} + r^2 \, \mathrm{d}\Omega \otimes \mathrm{d}\Omega \right) \tag{44}$$

$$\alpha(t) = (c/\lambda)^{1/2} t^{-1} \qquad c > 0 \qquad \lambda > 0. \tag{45}$$

For such a solution $A = 0$, $B = C = -\varepsilon/R_0^2 t^2$ and an invariant characterisation of the curvature is

$$R_{ab} * R^{ab} = \tfrac{3}{2}(c^2/t^4) * 1. \tag{46}$$

3. Conclusions

We have analysed a formulation of a self-coupled scalar interacting with gravity in terms of a non-Christoffel although metric compatible connection. In the absence of the self-interaction the theory may be related to either Brans–Dicke or Einstein–Klein–Gordon. When $\lambda \neq 0$ the action resembles a class of background $\lambda\alpha^4$ theories (Davis and Unwin 1981, Ford and Toms 1982) that have been investigated in curved space–time in the context of dynamical symmetry breaking. Indeed it may be of some interest to reinterpret such theories without assuming that gravity should be incorporated by adding to (7) the Einstein action. Our solutions (44) and (45) show that certain Robertson–Walker metrics solve the equations based on the action (7) exactly without an additional curvature term.

For $\lambda = 0$ the equivalence to the Brans–Dicke theory has been explicitly verified by recovering a class of known vacuum solutions. For $\lambda \neq 0$, new solutions have been derived including a Robertson–Walker universe with the topology $R \times S^3$. It is interesting to observe that for these solutions the scalar self-coupling $\lambda\alpha^4 * 1$ in the action decreases as an inverse power of the cosmic time t.

References

Blyth W F and Isham C J 1975 *Phys. Rev.* D **11** 768
Christensen S M and Duff M J 1980 *Nucl. Phys.* B **170** F51, 480
Davis P and Unwin S 1981 *Proc. R. Soc.* **3A 377** 147
Dereli T and Tucker R W 1982 *Phys. Lett.* **110B** 206
Ford L H and Toms D J 1982 *Dynamical symmetry breaking due to radiative corrections in cosmology, ICTP*/81/81-2
O'Hanlon J and Tupper B 1972 *Nuovo Cimento* B **7** 305
Parker L 1978 in *Recent Developments in Gravitation, Cargese* 1978 ed M Levy and S Deser (New York: Plenum)
Raychaudhuri A K 1979 *Theoretical Cosmology* (Oxford: Clarendon)